Irreducibility

• Irreducible
  – every state can be reached from every other state
  – For any i,j, exist an m₀, such that \( p_{ij}^{(m_0)} > 0 \)
    • i,j are communicate, if the above condition is valid
    • Irreducible: all states are communicate with each other
  – Absorbing state: \( p_{jj} = 1 \)
    • Reducible chain, i is an absorbing state
Recurrence

- $f_{jj}^{(n)}$: the probability that first returning to $j$ occurs $n$ steps after leaving $j$

- $f_{jj} := \sum_{n=1}^{\infty} f_{jj}^{(n)}$: the probability that the chain ever returns to $j$
  - $j$ is recurrent, if $f_{jj}=1$
  - $j$ is transient, if $f_{jj}<1$

- If $j$ is recurrent, mean recurrence time: $m_{jj} = \sum_{n=1}^{\infty} nf_{jj}^{(n)}$
  - If $m_{jj}<\infty$, $j$ is positive recurrent
  - If $m_{jj}=\infty$, $j$ is null recurrent
    - E.g., random walk return to origin $0, \frac{1}{2}, \frac{1}{4}, 1/8, 1/16, ...$
Periodicity

• Period of a state i
  – Greatest common divisor (gcd) \( \text{gcd}\{ n : p_{ii}^{(n)} > 0 \} \)

• Aperiodic Markov chain
  – A DTMC (discrete time Markov chain) is aperiodic iff all states have period 1

• Example
  – period is 2, for DTMC with \( P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \)
  – Limiting of \( P^n \) does not exist, \( P^{2n} \) does.
Ergodicity

• Ergodic Markov chain
  – positive recurrent, irreducible, aperiodic
  – finite, irreducible, aperiodic

• Physical meaning
  – If ergodic, time average = ensemble average
    • For some moments, process is ergodic
    • Fully ergodic: for all moments, process is ergodic
  – See page 35 of Gross’ book
Limiting distribution

• If a finite DTMC is aperiodic and irreducible, then \( \lim_{n \to \infty} p_{ij}^{(n)} \) exists and is independent of \( i \)

• 3 types of probabilities
  – Limiting prob. \( l_j := \lim_{n \to \infty} p_{ij}^{(n)} \)
  – Time-average prob. \( p_j := \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} 1(X(n) = j) \)
  – Stationary prob. \( \pi_j, \) s.t. \( \pi P = \pi \)
Limiting probability

• Limiting probability of $i$
  – The limiting situation to enter $i$, may not exist (consider periodic case)

• Time-average probability of $i$
  – % of time staying at state $i$

• Stationary probability
  – Given the initial state is chosen according to $\pi$, all the future states follows the same distribution
Example, binary communication channel

- Limiting prob.

\[ l_j = \begin{cases} 
DNE, & \text{if } a = b = 1 \\
\frac{b}{a+b}, & \frac{a}{a+b}
\end{cases} \]

- Time-average prob.

\[ p_j = \left( \frac{b}{a+b}, \frac{a}{a+b} \right) \]

- Stationary prob.

- Solve the equation \( \pi P = \pi \) and \( \pi e = 1 \)

- Conclusion

- Limiting prob. may not exist, otherwise they are all identical

- \( \pi \) and \( p \) are always the same
Limiting probability

• Relations of 3 probabilities
  If the limiting probability exists, then these 3 probabilities match!
  – If ergodic, limiting prob. exists and 3 prob. match

• How to solve?
  \[ \pi P = \pi, \quad \pi e = 1 \quad Pe = e \]
  – \( \pi \) is the eigen vector of matrix \( P \) with eigen value 1
  – balance equation
Example, discrete time birth-death process

• Model the dynamic of population

• Parameters
  – Increase by 1 with a birth prob. \( b \)
  – Decrease by 1 with a death prob. \( d \)
  – Stay the same state with prob. \( 1-b-d \)

• Chalk write
  – Write the transition probability matrix \( P \)
  – Solve the equation for stationary prob.
  – Understand the balance equation
Part II: continuous time Markov chain (CTMC)

• Continuous time discrete state Markov process
• Definition (Markovian property)
  – $X(t)$ is a CTMC, if for any $n$ and any sequence $t_1 < t_2 < ..., < t_n$, we have

$$P[X(t_n) = j \mid X(t_{n-1}) = i_{n-1}, X(t_{n-2}) = i_{n-2}, ..., X(t_1) = i_1] = P[X(t_n) = j \mid X(t_{n-1}) = i_{n-1}]$$
Exponentially distributed sojourn time of CTMC

• Sojourn time at state $i$, $\tau_i$, $i=1,2,...,S$, is exponentially distributed
  – Proved with Markovian property

\[
P[\tau_i > s + t \mid \tau_i > s] = P[\tau_i > t] \quad P[\tau_i > s + t, \tau_i > s] = P[\tau_i > s]P[\tau_i > t]
\]

\[
P[\tau_i > s + t] = P[\tau_i > s]P[\tau_i > t]
\]

\[
\frac{dP[\tau_i > s + t]}{ds} = -f_{\tau_i}(s)P[\tau_i > t]
\]

\[
\frac{dP[\tau_i > t]}{P[\tau_i > t]} = -f_{\tau_i}(0)ds\mid_{s \to 0} \quad \ln P[\tau_i > t] = -f_{\tau_i}(0)t \quad P[\tau_i > t] = e^{-f_{\tau_i}(0)t}
\]

\[
f_{\tau_i}(t) = f_{\tau_i}(0)e^{-f_{\tau_i}(0)t}
\]

Sojourn time is exponentially distributed! $f_{\tau_i}(0)$ is transition rate of state $i$
Basic elements of CTMC

• State space: $S = \{1, 2, \ldots, S\}$, assume it is discrete and finite for simplicity

• Transition prob. $p_{ij}(s, t) := P[X(t) = j \mid X(s) = i], \quad s \leq t$
  
  – In matrix form $P(s, t) := \begin{bmatrix} p_{ij}(s, t) \end{bmatrix}$
  
  – If homogeneous, denote $P(t) := P(s, s + t)$
    
    • $P(t)$ is independent of $s$ (start time)
  
  – Chapman Kolmogorov equation
    
    $P(s, t) = P(s, u)P(u, t), \quad s \leq u \leq t$
Infinitesimal generator (transition rate matrix)

• Differentiation of C-K equation, let $u \rightarrow t$

$$\frac{\partial P(s,t)}{\partial t} = P(s,t)Q(t), \quad s \leq t$$

$$Q(t) = \lim_{\Delta t \to 0} \frac{P(t,t+\Delta t) - I}{\Delta t}$$

• $Q(t)$ is called the infinitesimal generator of transition matrix function $P(s,t)$ since

  - Its elements are

  $$q_{ii}(t) = \lim_{\Delta t \to 0} \frac{p_{ii}(t,t+\Delta t) - 1}{\Delta t} \leq 0$$

  $$q_{ij}(t) = \lim_{\Delta t \to 0} \frac{p_{ij}(t,t+\Delta t)}{\Delta t} \geq 0, \quad i \neq j$$

  $$\sum_{j=1}^{s} q_{ij}(t) = 0, \quad \text{for all } i$$

  $$Q(t)e = 0$$

  - relation between transition rate $q_{ij}$ and probability $p_{ij}$

  - $-q_{ii}$ is the rate flow out of $i$, $q_{ij}$ is the rate flow into $j$
Infinitesimal generator

- If homogeneous, denote $Q=Q(t)$
- Derivative of transition probability matrix

\[
\frac{\partial P(t)}{\partial t} = P(t)Q
\]

- With initial condition $P(0)=I$, we have

\[
P(t) = e^{Qt} \quad \text{where } e^{Qt} := I + Qt + \frac{1}{2!} Q^2 t^2 + \frac{1}{3!} Q^3 t^3 + \ldots
\]

- Infinitesimal generator (transition rate matrix) $Q$ can generate the transition probability matrix $P(t)$
State probability

- $\pi(t)$ is the state probability at time $t$
- We have
  \[ \pi(t) = \pi(0)P(t) = \pi(0)e^{Qt} \]
- Derivative operation
  \[ \frac{d\pi(t)}{dt} = \pi(0)e^{Qt}Q = \pi(t)Q \]
- Decompose into elements
  \[ \frac{d\pi_i(t)}{dt} = \pi_i(t)q_{ii} + \sum_{j\neq i} \pi_j(t)q_{ji} \]
Steady state probability

• For ergodic Markov chain, we have

\[ \lim_{t \to \infty} \pi_j(t) = \pi_j = \lim_{t \to \infty} p_{ij}(t) \]

• Balance equation

\[ \frac{d\pi_i(t)}{dt} = \pi_i(t)q_{ii} + \sum_{j \neq i} \pi_j(t)q_{ji} = 0 \]

• In matrix

\[
\begin{align*}
\pi Q &= 0 \\
\pi e &= 1 \\
Q e &= 0
\end{align*}
\]

– Comparing this with discrete case
Birth-death process

• A special Markov chain
  – Continuous time: state \( k \) transits only to its neighbors, \( k-1 \) and \( k+1 \), \( k=1,2,3,... \)
  – model for population dynamics
  – basis of fundamental queues

• state space, \( S=\{0,1,2,...\} \)

• at state \( k \) (population size)
  – Birth rate, \( \lambda_k \), \( k=0,1,2,... \)
  – Death rate, \( \mu_k \), \( k=1,2,3,... \)
Infinitesimal generator

• Element of $Q$
  $- q_{k,k+1} = \lambda_k , k=0,1,2,...$
  $- q_{k,k-1} = \mu_k , k=1,2,3,...$
  $- q_{k,j} = 0, |k-j|>1$
  $- q_{k,k-1} = -\lambda_k - \mu_k$

• Chalk writing: tri-diagonal matrix $Q$
State transition probability analysis

• In $\Delta t$ period, transition probability of state $k$
  – $k \to k+1$: $p_{k,k+1}(\Delta t) = \lambda_k \Delta t + o(\Delta t)$
  – $k \to k-1$: $p_{k,k-1}(\Delta t) = \mu_k \Delta t + o(\Delta t)$
  – $k \to k$: $p_{k,k}(\Delta t) = [1 - \lambda_k \Delta t - o(\Delta t)][1 - \mu_k \Delta t - o(\Delta t)]$
    \[ = 1 - \lambda_k \Delta t - \mu_k \Delta t + o(\Delta t) \]
  – $k \to \text{others}$: $p_{k,j}(\Delta t) = o(\Delta t), \quad |k-j| > 1$
State transition equation

- State transition equation

\[ \pi_k(t + \Delta t) = \pi_k(t)p_{k,k}(\Delta t) + \pi_{k-1}(t)p_{k-1,k}(\Delta t) + \pi_{k+1}(t)p_{k+1,k}(\Delta t) + o(\Delta t) \]

\( k=1,2,… \)

Case of \( k=0 \) is omitted, for homework.
State transition equation

• State transition equation:

\[ \pi_k(t + \Delta t) = \pi_k(t) - (\lambda_k + \mu_k) \Delta t \pi_k(t) + \lambda_{k-1} \Delta t \pi_{k-1}(t) + \mu_{k+1} \Delta t \pi_{k+1}(t) + o(\Delta t) \]

• With derivative form:

\[
\frac{d\pi_k(t)}{dt} = - (\lambda_k + \mu_k) \pi_k(t) + \lambda_{k-1} \pi_{k-1}(t) + \mu_{k+1} \pi_{k+1}(t), \quad k = 1, 2, \ldots
\]

\[
\frac{d\pi_0(t)}{dt} = - \lambda_0 \pi_0(t) + \mu_1 \pi_1(t), \quad k = 0
\]

– This is the equation of \( d\pi = \pi Q \)

– Study the transient behavior of system states

\[ \pi(t) = \pi(0) e^{Qt} \]
Easier way to analyze birth-death process

• State transition rate diagram

  - This is transition rate, not probability
    • If use probability, multiply dt and add self-loop transition
  - Contain the same information as Q matrix

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State transition rate diagram

- Look at state $k$ based on state transition rate diagram
  - Flow rate into state $k$
    \[ I_k = \lambda_{k-1} \pi_{k-1}(t) + \mu_{k+1} \pi_{k+1}(t) \]
  - Flow rate out of state $k$
    \[ O_k = (\lambda_k + \mu_k) \pi_k(t) \]
  - Derivative of state probability
    \[
    \frac{d\pi_k(t)}{dt} = I_k - O_k = \lambda_{k-1} \pi_{k-1}(t) + \mu_{k+1} \pi_{k+1}(t) - (\lambda_k + \mu_k) \pi_k(t)
    \]
State transition rate diagram

• Look at a state, or a set of neighboring states
  – Change rate of state probability = incoming flow rate – outgoing flow rate

• Balance equation
  – For steady state, i.e., the change rate of state probability is 0, then we have balance equation
    incoming flow rate = outgoing flow rate,
    to solve the steady state probability
  – This is commonly used to analyze queuing systems, Markov systems
Pure birth process

• Pure birth process
  – Death rate is $\mu_k = 0$
  – Birth rates are identical $\lambda_k = \lambda$, for simplicity

• We have the derivative equation of state prob.

$$
\frac{d\pi_k(t)}{dt} = \lambda\pi_{k-1}(t) - \lambda\pi_k(t), \quad k = 1, 2, ...
$$

$$
\frac{d\pi_0(t)}{dt} = -\lambda\pi_0(t), \quad k = 0
$$

– assume initial state is at $k=0$, i.e., $\pi_0(0)=1$, we have

$$
\pi_0(t) = e^{-\lambda t}
$$
Pure birth process

- For state $k=1$

\[
\frac{d\pi_1(t)}{dt} = -\lambda \pi_1(t) + \lambda e^{-\lambda t}
\]

we have \( \pi_1(t) = \lambda t e^{-\lambda t} \)

- Generally, we have

\[
\pi_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t \geq 0, k = 0, 1, 2, \ldots
\]

– The same as Poisson distribution!
Pure birth process and Poisson process

• Pure birth process is exactly a Poisson process
  – Exponential inter-arrival time with mean $1/\lambda$
  – $k$, number of arrivals during $[0,t]$

$$P_k = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, ...$$

– Z-transform of random variable $k$
  • To calculate the mean and variance of $k$
– Laplace transform of inter-arrival time
  • To calculate the mean and variance of inter-arrival time
Arrival time of Poisson process

• X is the time that exactly having k arrivals
  – X = t_1 + t_2 + ... + t_k
  – t_k is the inter-arrival time of the kth arrival
  – X is also called the k-Erlang distributed r.v.

• The probability density function of X
  – Use convolution of Laplace transform

\[ f_X(x) = \lambda \cdot \frac{(\lambda x)^{k-1}}{(k-1)!} e^{-\lambda x} \]

\[ X*(s) = \left[ A*(s) \right]^k = \left[ \frac{\lambda}{\lambda + s} \right]^k \]

Look at table of Laplace transform